On a new class of the commutative subalgebras of the maximal Gel'fand-Kirillov dimension in the universal enveloping algebra of a simple Lie algebra

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Abstract

New commutative subalgebras of the maximal Gel'fand-Kirillov dimension in the universal enveloping algebras of the classical Lie algebras $gl(n)$ and $so(n)$ are constructed. In the case of $sp(n)$ Gel'fand-Tsetlin algebra is extended to a maximally commutative one.

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1   Introduction

Commutative subalgebras of maximal degree of transcendency (it is known [1] that for commutative associative algebra the degree of transcendency coinsides with Gel'fand- Kirillov dimension) in the universal enveloping algebras are very important both from the point of view of the representation theory and the theory of quantization of completely integrable systems. In the framework of the latter these subalgebras play the role of the algebras of quantum integrals of motion. In both theories a possibility to diagonalize the generators of these algebras is very important. It may yield the basis in the space of representation which consists of common eigenvectors of all generators of these algebras. Examples of such the construction are known: so-called Gel'fand-Tsetlin algebras consisting of the Casimir elements of a chain of embedded subalgebras. Bases of their common eigenvectors are celebrated Gel'fand-Tsetlin bases [2]-[3]. Except for the special case of [4] there were no commutative subalgebras of the maximal degree of transcendency other than Gel'fand-Tsetlin algebras known in the universal enveloping algebras of the simple Lie algebras.

In this paper we find new commutative subalgebras in the universal enveloping algebras $U(g)$ of the simple Lie algebras g that have maximal degree of transcendency and include any given commutative subalgebra $C_{g_A} \subset U(g_A)$ for the next pairs (g, g_A) of Lie algebras and their subalgebras: $(gl(n), gl(n-2) + gl(2)), (so(n), so(n-2) + so(2))$ and $(sp(n), sp(n-1) + gl(1))$. Idea that lead to the discovery of such the algebras comes out from the theory of integrable dynamical systems. As it is known [5]- [6] there is sufficiently large (multiparametric) class

of commutative with respect to the Lie-Poisson brackets subalgebras in the algebra of the polinomial functions $P(g^*)$ on the dual to the Lie algebra space g^*. Algebra of polinomial functions on g^* can be viewed as a graded algebra of the filtered algebra $U(g)$ and the Lie-Poisson brackets as the graded part of the commutator in $U(g)$. Equality to zero of the Lie-Poisson brackets of the elements of $P(g^*)$ is equivalent to the equality to zero of the higher order in the elements of g part of the commutator (but not of the whole commutator!) of the correspondent elements of $U(g)$. This means that, in general, commutative with respect to the Lie-Poisson brackets subalgebra in $P(g^*)$ may not have commutative counterpart in $U(g)$. But if a commutative subalgebra in $P(g^*)$ is an algebra of integrals of some dynamical system of the Euler-Arnold type then from the physical consideration it follows, that corresponding quantum dynamical system should have the same algebra of the integrals, meaning that counterpart of commutative algebra in $P(g^*)$ have also to be commutative subalgebra in $U(g)$. For the case when underlying dynamics is the dynamics of the generalized rigid body the corresponding algebra of the integrals coincides with the Mishchenko-Fomenko algebra [5]-[6]. Although it was conjectured by Vinberg [7] that the Mishchenko-Fomenko algebra could be "lifted" to the universal enveloping algebra without the lost of the commutativity, he managed to prove this fact only for its special subalgebra, characterized as having a second order in the elements of the Lie algebra g. We prove (Theorems 1-2) the commutativity in $U(g)$ of the other analogue of the subalgebra of the Mishchenko-Fomenko algebra. Contrary to the case considered by Vinberg, our subalgebra in the $U(g)$ is not "homogeneous" in the elements of g but is linear in the "parameters of the shift" $(A)_{i,j}$, i.e. additional parameters

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which are important ingradients in the definition of the Mishchenko-Fomenko algebras [5]-[6] [1]. We use it to construct commutative subalgebra in $U(g)$, including its generators, the Casimir elements of $U(g)$ and generators of a maximally commutative subalgebra in $U(g_A)$, where g_A is a stabilizer of a fixed matrix A in g, in the set of generators of the commutative subalgebra in $U(g)$. Using results of [9], [10], we prove (Theorem 3) that in the cases when "shift" matrix A is of the second rank these commutative subalgebras in $U(g)$ have the maximal degree of transcendency equal to $(\dim g + \text{ind} g)/2$.

Importance of these commutative subalgebras is explained by their maximal degree of transcendency and the fact that they contain the maximal number of generators that commute with the described above Lie subalgebras $g_A \in g$. These both facts together mean that they should separate all vectors of the isomorphic (multiple) representations of the Lie subalgebra g_A that emerge after a restriction of a representation of the Lie algebra g to its subalgebra g_A. [2]

The structure of our article is following: in section 2 we prove commutativity of the described above subalgebras in $U(g)$, in section 3 we prove the maximallity of the degree of transcendency for the constructed commutative subalgebras, and in the last sections we give the proofs of the main theorems.

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[1] For definition of the Mishchenko-Fomenko algebras see also section 6.

[2] See [4] for a "physical" example of the representations of $su(2,2) \subset gl(4, \mathbf{C})$.

2 Commutativity

Here we will present some commutative subalgebras in the universal enveloping algebra, that will be used in the next section to construct maximal commutative subalgebras. All considered algebras in this section are over a field $\mathbf{K}$ of characteristic zero.

2.1 Let us consider the case of $gl(n)$ algebra first. Let $X_{i,j}$, $i, j \in I$ where $I = (1, 2, ..., n)$, be the basis in this algebra with the commutation relations:

$$[X_{i,j}, X_{k,l}] = \delta_{k,j} X_{i,l} - \delta_{i,l} X_{k,j}.$$

Universal enveloping algebra $U(gl(n))$ consists of formal polinomials in the elements $X_{i,j}$. Let us define the following elements of $U(gl(n))$:

$$(X^M)_{i,j} = \sum_{i_1,...i_{M-1} \in I} X_{i,i_1} X_{i_1,i_2}...X_{i_{M-1},j}.$$

It is known [8], that the elements: $(X^M) = \sum_{i \in I} (X^M)_{i,i}$, $M \in (1, 2, ..., n)$ generate the center of the universal enveloping algebra. For arbitrary numerical matrix $A \in \mathrm{Mat}(n, \mathbf{K})$ and positive integer M we can introduce the following elements of $U(g)$:

$$(AX^M) = \sum_{i,j \in I} A_{j,i} (X^M)_{i,j}.$$

They are "quantum" analogues of the generators of the subalgebra of the Mishchenko-Fomenko algebra linear in the "shift" parameters $A_{j,i}$.

Let $g_A \subset gl(n)$ be the subspace in g spanned over $\mathbf{K}$ by the elements $(BX) = \sum_{i,j \in I} B_{j,i} X_{i,j}$, where numerical matrix B belongs to the stabilizer of A in $\mathrm{Mat}(n, \mathbf{K})$, i. e. $[B, A] = 0$. It is easy to show that $[(BX), (CX)] = ([B, C]X)$, i. e. that g_A is a subalgebra in g isomorphic to the stabilizer of matrix A in $\mathrm{Mat}(n, \mathbf{K})$.

Now we are able to formulate the theorem.

Theorem 1. *Let F_A be a subalgebra in $U(gl(n))$ generated by (X^N) and (AX^M), $M, N \in \mathbf{Z}_+$. Then*

(i) F_A is a commutative subalgebra in $U(gl(n))$.

(ii) g_A centralizes F_A in $gl(n)$.

Idea of proof. Proof of the statement (ii) of Theorem 1 is based on the fact that elements $(X^M)_{i,j}$ are tensorial [8] . We prove statement (i) using the fact that higher order in the elements of $gl(n)$ part of the commutator $[(AX^M), (AX^N)]$ is trivial due to the commutativity of (AX^M) and (AX^N) in $P(gl(n)^*)$, the observation that the other part of the commutator could be written in the form of the sum of the communators of (AX^K) and (AX^L) where $K < M$ and the obvious fact that $(AX) \in g_A$. See section 4 for detailed proof.

2.2 In this subsection we consider the case of the algebras $so(n)$ and $sp(n)$ which will be denoted by g. It will be convenient to treat them uniformly, so that the algebras $so(2n)$ and $sp(n)$ be considered as subalgebras of $gl(2n)$ and algebra $so(2n+1)$ as a subalgebra of $gl(2n+1)$ respectively. Let $X_{i,j}, i, j \in I$ be the basis in these algebras, $I = (-n, ..., -1, 0, 1, ..., n)$ for the case of $so(2n+1)$, $I = (-n, ..., -1, 1, ..., n)$ for the case of $sp(n)$ and $so(2n)$, with the commutation relations:

$$[X_{i,j}, X_{k,l}] = \delta_{k,j} X_{i,l} - \delta_{i,l} X_{k,j} + \epsilon_i \epsilon_j (\delta_{j,-l} X_{k,-i} - \delta_{k,-i} X_{-j,l})$$

and additional property $X_{i,j} = -\epsilon_i \epsilon_j X_{-j,-i}$ (here $\epsilon_j = 1$ for the case $g = so(n)$ and $\epsilon_j =$ sgn j for the case $g = sp(n)$). Universal enveloping algebra $U(g)$ consists of formal polinomials in the elements $X_{i,j}$. Let $(X^M)_{i,j}$, (X^M), (AX^M) be elements of $U(g)$ and

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$F_A \subset U(g)$, $g_A \subset g$ be subalgebras of $U(g)$ defined like in previous subsection. Then the following theorem analogous to Theorem 1 holds:

Theorem 2. *If matrix A satisfies condition $A_{i,j} = \pm\epsilon_i\epsilon_j A_{-j,-i}$ then*

(i) F_A is a commutative subalgebra in $U(g)$.

(ii) g_A coincides with the centralizer of F_A in g.

Idea of proof. Proof of Theorem 2 is based on the same idea as the proof of Theorem 1 (see the end of the previous subsection). See section 5 for the detailed proof.

Remark 1. The type of the realization of algebras $so(n)$ and $sp(n)$ used in this section is not of any importance. For example we could use standard realization of the algebra $so(n)$ with the basis $X_{i,j}, i, j \in (1, 2, ..., n)$, $X_{i,j} = -X_{j,i}$ that satisfies the commutation relations:

$$[X_{i,j}, X_{k,l}] = \delta_{k,j}X_{i,l} - \delta_{i,l}X_{k,j} + \delta_{j,l}X_{k,i} - \delta_{k,i}X_{j,l}.$$

Proof of Theorem 2 in this standard realization is the same as in the previous case. Important in the above construction is the treatment of the $so(n)(sp(n))$ algebras as subalgebras of $gl(n)(gl(2n))$ that are stable under the action of the involutive automorphism σ. For elements $(AX^N) \in U(gl(n))$ to remain commutative after the restriction to the universal enveloping algebra of these subalgebras for arbitrary realization of the automorphism σ, we have to require equality $\sigma(A) = \pm A$, which in our case has the form: $A_{i,j} = \pm\epsilon_i\epsilon_j A_{-j,-i}$.

Remark 2. If $A_{i,j} = +\epsilon_i\epsilon_j A_{-j,-i}$ ($\sigma(A) = -A$ in general) then "quantum integrals" obtained above correspond to the classical integrals of "normal series" [6]. In the next sections we will consider only subalgebras constructed with the help of the matrix A, that satisfies the condition $A_{i,j} = -\epsilon_i\epsilon_j A_{-j,-i}$, meaning that numerical matrix A belong to the

same subalgebra of $gl(n)$ as the considered Lie algebra.

3 Maximality

3.1 Algebras F_A constructed in the previous section for all classical Lie algebras g obviously have degrees of transcendency not higher than 2 rank g -1. Nevertheless statement (ii) of Theorem 1 and Theorem 2 enables us to extend them to the commutative algebras having maximal degree of transcendency. Indeed, according to this statement we can use in the construction of maximal commutative subalgebra in $U(g)$ not only generators of F_A but also some commutative subalgebra in $U(g_A)$. In the case of sufficiently singular matrix A we may hope to find in $U(g_A)$ "large " commutative subalgebra, that will "complete" F_A to the one having maximal degree of transcendency. Indeed, the following theorem is true.

Theorem 3. *Let g be simple matrix Lie algebra over the field of real or complex numbers, $A \in g$ be a constant semisimple matrix of the second matrix rank, g_A and F_A be as in section 2. Let C_{g_A} be some commutative subalgebra in $U(g_A)$, $F(g, C_{g_A})$ be a commutative subalgebra in $U(g)$ generated by F_A and C_{g_A}.*

If deg.tr. $C_{g_A} = (\dim g_A + \operatorname{ind} g_A)/2$ *then* deg.tr. $F(g, C_{g_A}) = (\dim g + \operatorname{ind} g)/2.$ [3]

Idea of proof. To prove this Theorem we use commutativity of the algebras F_A and C_{g_A}, provided by Theorems 1-2 and the assumption of Theorem 3, and the results of Vinberg [7] which enables us to consider problem of finding the degree of transcendency of our subalgebra

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[3]The abbreviation *deg.tr.* means "the degree of transcendency ".

in $P(g^*)$ instead $U(g)$. Hence the problem of finding the degree of transcendency is reduced to the problem of finding of the number of functionally independent generators of the algebra $F(g, C_{g_A})$ in $P(g^*)$. That permits us to use results of [9] about a number of functionally independent generators of the Mishchenko-Fomenko algebra in the case of a singular covector of the shift and results of [10] concerning their explicit form. See section 6 for detailed proof.

Remark 1. The condition that numerical matrix A is of matrix rank 2 is essential. Indeed, if its matrix rank exceeds 2 then from the dimensional considerations follows that $\mathrm{deg.tr.}F(g, C_{g_A}) < (\dim g + \mathrm{ind} g)/2$ for arbitrary commutative algebra $C_{g_A} \in U(g_A)$ [10]. On the contrary, it is possible to show that in the case of matrix A of rank 1 elements (AX^N) are functionally dependent on the Casimir elements of $U(g)$ and $U(g_A)$. So in the last case they will not contribute to the degree of transcendency of the constructed commutative subalgebras.

This theorem enables us to construct the commutative algebras of maximal degree of transcendency in the universal enveloping algebra of an arbitrary simple Lie algebra. Indeed, if there is no known commutative subalgebra in $U(g_A)$ with the maximal Gel'fand-Kirillov dimension, as in the case of $g = sp(n)$, $g_A = sp(n-1) + gl(1)$, we can use the same method of constructing commutative subalgebras in the universal enveloping algebra for each simple component of the g_A. Repeating this procedure we at last will come to the situation when all components of stabilizer satisfy conditions of Theorem 3 (for example when they have rank one as a simple Lie algebras).

In the general situation in order to obtain "large" commutative subalgebras in the uni-

versal enveloping algebra one can combine "method of the chain of subalgebras" with the method of " the shift of the argument", or to be more precise, its special case connected with special singular "constant covector A", varying its degree of singularity from 1 (except the case of symplectic algebra) to 2 for each successive pair of an algebra and its subalgebra in the chain. In the next subsections we will consider in the detail this construction for all classical simple (reductive) Lie algebras over the field $\mathbf{C}$ of a complex numbers. We will use the following easily proved

Lemma 1. *Let $A \in g \subset Mat(n, \mathbf{K})$ be numerical semisimple matrix of the second matrix rank and the basic field $\mathbf{K}$ be the field of the complex numbers. If g is one of the classical simple Lie algebras of the type A_n, B_n, C_n, D_n then g_A will contain simple Lie subalgebra of the type $A_{n-2}, B_{n-1}, C_{n-1}, D_{n-1}$ correspondently.*

3.2 Let us start with the simplest case $g = gl(n)$ (or, equivalently, $g = sl(n)$). The next corollary of Theorem 3 and Lemma 2 holds.

Corollary 1. *Let us consider the chain of subalgebras in $gl(n)$ of the type*

$$gl(n) \supset gl(n - k_1) \supset gl((n - k_1) - k_2)) \supset ... \supset gl(1).$$

If $k_i = 1, 2$ for any index i, then one can associate with this chain a commutative subalgebra of the maximal degree of transcendency (equal to $n(n+1)/2$) in $U(gl(n))$ that consists of the Casimir elements of the each subalgebra in the chain, and if subalgebra next to $gl(i)$ in the chain is $gl(i-2)$, elements of $U(gl(n))$ of the type $(A_{(i)} X_{(i)}^N), N < n$. Index (i) here means that $X_{(i)}$ belongs to corresponding subalgebra $gl(i)$, $A_{(i)} \in Mat(i)$ is complex semisimple matrix of the second matrix rank.

Example. Let $g = gl(n)$; semisimple rank two matrix A be of the type $A = diag(a_1, a_2, 0, ..., 0)$, where $a_i \in \mathbf{C}, a_i \neq o$. Then $g_A = gl(n-2) + gl(1) + gl(1)$, if $a_1 \neq a_2$ or $g_A = gl(n-2) + gl(2)$ if $a_1 = a_2$. In both cases we can construct commutative subalgebras in $U(gl(n))$ with degree of transcendency equal to $n(n+1)/2$. For its independent generators we can take:

1) $n = \text{rank } gl(n)$ independent Casimir elements of type (X^k), $k \in (1, 2, ..., n)$.

2) $n - 1$ independent elements of $U(gl(n))$ of the type (AX^m), $m \in (1, 2, ...n - 1)$.

3) $(n-1)(n-2)/2$ generators of the Gel'fand-Tsetlin algebra of $gl(n-2)$.

In the second case only we can use instead of the Gel'fand-Tsetlin algebra of $gl(n-2)$ the Gel'fand-Tsetlin algebra of $gl(n-2) + gl(2)$. In this case we have to exclude from the generators of the whole commutative algebra generators $(AX), (AX^2)$ which would be no more algebraically independent. In the special case $n = 4$ such the commutative algebra was considered by Yao [4] [4].

Remark 1. If $k_1 = 2$ and complex semisimple matrix $A = A_{(n)}$ of rank two has two equal non zero eigenvalues then all commutative subalgebras, constructed in the Corollary 1, centralize Lie subalgebra $g_A = gl(n) + gl(2)$.

Remark 2. It is interesting to notice that for all commutative subalgebras constructed in the Corollary 1, number of the generators, that belong to the given filtered component of the universal enveloping algebra, is fixed and is the same as in the Gel'fand-Tsetlin algebra.

3.3 The case $g = so(n)$ repeats almost completely the case of the $gl(n)$. The following

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[4]In the case considered by Yao additionally to the Casimir elements of $gl(4)$ and Gel'fand-Tsetlin algebra of $gl(2) + gl(2)$ only one $Gl(2) \times Gl(2)$ invariant element of $U(gl(4))$ was needed.

corollary of Theorem 3 and Lemma 1 holds.

Corollary 2. *Let us consider the chain of subalgebras in* $so(n)$ *of the type*

$so(n) \supset so(n - k_1) \supset so((n - k_1) - k_2)) \supset ... \supset so(2).$

If $k_i = 1, 2$ *for any index* i, *then one can associate with this chain the commutative subalgebra in* $U(so(n))$ *of the maximal degree of transcendency equal to* $((n-1)n/2+[n/2])/2$, *that consists of the Casimir elements of the each subalgebra in the chain and, if a subalgebra next to* $so(i)$ *in the chain is* $so(i - 2)$, *elements of* $U(so(n))$ *of the type* $(A_{(i)}X_{(i)}^N)$, $N < n$,*N is odd. Here index* (i) *means that* $X_{(i)}$ *belongs to the corresponding subalgebra* $so(i)$ *and* $A_{(i)} \in so(i)$ *is complex semisimple matrix of the second matrix rank.*

Remark 1. If $k_1 = 2$ then all the commutative subalgebras constructed above centralize $so(n - 2) + so(2)$.

Remark 2. Contrary to the case the $g = gl(n)$ commutative subalgebras corresponding to the different chains of the subalgebras have different number of the generators, belonging to the given filtered component of the universal enveloping algebra. In particular, if we take $k_i = 2$, for all indices i, we'll obtain commutative subalgebra in the universal enveloping algebra that contains $[n/2]$ linear in elements of $so(n)$ generators. To make comparison we remind that Gel'fand-Tsetlin algebra for this case has only one generator linear in the elements of the $so(n)$.

3.4 The case $g = sp(n)$ differs from the considered above cases of $gl(n)$ and $so(n)$. In this case neither for $sp(n)$ nor for any of its subalgebra $sp(n - k)$, $k \in (1, 2, ..., n - 2)$ Gel'fand -Tsetlin algebra is of maximal degree of transcendency equal to $(n - k)(n - k + 1)$.

That is why our construction gives only one commutative subalgebra in the $U(sp(n))$ of the maximal degree of transcendency. This commutative subalgebra will be the extended Gel'fand -Tsetlin algebra . This is expressed in the following Corollary:

Corollary 3. *Let us consider the following chain of subalgebras in $sp(n)$*

$$sp(n) \supset sp(n-1) \supset sp(n-2) \supset ... \supset sp(1) \supset gl(1).$$

A commutative subalgebra in $U(sp(n))$ of the maximal degree of transcendency equal to $n(n+1)$, that consists of the Casimir elements of the each subalgebra in the chain and elements of $U(sp(n))$ of the type $(A_{(i)}X_{(i)}^k)$, k is odd, $k < 2i$, is associated with this chain. Index (i) means that $X_{(i)}$ belongs to correspondent subalgebra $sp(i)$ and $A_{(i)} \in sp(i)$ is complex semisimple matrix of the second rank.

4 Proof of Theorem 1.

It is easy to verify [8] that elements $(X^M)_{i,j}$ are "tensorial" :

$$[X_{i,j}, (X^M)_{k,l}] = \delta_{k,j}(X^M)_{i,l} - \delta_{i,l}(X^M)_{k,j}.$$

This yields the validity of the statement (ii). Indeed, from the tensorial nature of the elements $(X^N)_{i,j}$ follows that $[(BX),(AX^N)] = ([A,B]X^N)$ Hence (BX) and (AX^N) commute if and only if $[A,B] = 0$, i. e. if $(BX) \in g_A$. This proves (ii).

Now we will prove part (i) of the theorem. It is evident that Casimir elements of the type (X^K) commute among themselves and with all elements of $U(g)$. So to prove the statement it is necessary to show that elements of the type (AX^M) and (AX^N) commute. We will use

the following Proposition.

Proposition 1. $[(X^M)_{i,j}, (X^N)_{k,l}] = \sum\limits_{S=1}^{M} ((X^{M+N-S})_{i,l}(X^{S-1})_{k,j} - (X^{S-1})_{i,l}(X^{N+M-S})_{k,j}).$

Proof. Using the Leibnitz rule we obtain

$$[(X^M)_{i,j}, (X^N)_{k,l}] = \sum\limits_{P=1}^{M} \sum\limits_{u,v=1}^{n} ((X^{P-1})_{i,u}[X_{u,v}, (X^N)_{k,l}](X^{M-P})_{v,j}).$$

Using tensorial nature of the elements $(X^N)_{k,l}$ we will obtain the statement of the proposition.

Now we are able to prove (i). We will use inductive proof. Let us prove that from $[(AX^K), (AX^N)] = 0$, for all $N \in \mathbf{Z}_+, K < M - 1$ follows that $[(AX^M), (AX^N)] = 0$, for all $N \in \mathbf{Z}_+$. This inductive assumption will be proved by the following Proposition:

Proposition 2. $[(AX^M), (AX^N)] = \sum\limits_{S=1}^{M} \sum\limits_{P=1}^{S-1} [(AX^{P-1}), (AX^{M+N-P-1})].$

Proof. Transforming the left-hand-side of the equality we obtain

$$\sum\limits_{i,j,k,l \in I} A_{j,i} A_{l,k} [(X^M)_{i,j}, (X^N)_{k,l}] = \sum\limits_{i,j,k,l \in I} A_{j,i} A_{l,k} \sum\limits_{S=1}^{M} (X^{M+N-S})_{i,l}(X^{S-1})_{k,j} - (X^{S-1})_{i,l}(X^{N+M-S})$$

$$= \sum\limits_{i,j,k,l \in I} A_{j,i} A_{l,k} (\sum\limits_{S=1}^{M} (X^{M+N-S})_{i,l}(X^{S-1})_{k,j} - (X^{N+M-S})_{k,j}(X^{S-1})_{i,l} - [(X^{S-1})_{i,l}, (X^{N+M-S})_{k,j}])$$

$$= \sum\limits_{S=1}^{M} ((AX^{S-1}AX^{M+N-S}) - (AX^{S-1}AX^{M+N-S}) - \sum\limits_{i,j,k,l \in I} A_{j,i} A_{l,k} [(X^{S-1})_{i,l}, (X^{N+M-S})_{k,j}])$$

$$= -\sum\limits_{S=1}^{M} \sum\limits_{i,j,k,l \in I} A_{j,i} A_{l,k} [(X^{S-1})_{i,l}, (X^{N+M-S})_{k,j}] = -\sum\limits_{i,j,k,l \in I} A_{j,i} A_{l,k} (\sum\limits_{S=1}^{M} \sum\limits_{P=1}^{S-1} (X^{M+N-P-1})_{i,j} \cdot$$

$$\cdot (X^{P-1})_{k,l} - (X^{P-1})_{i,j}(X^{N+M-P-1})_{k,l}) = \sum\limits_{S=1}^{M} \sum\limits_{P=1}^{S-1} [(AX^{P-1}), (AX^{M+N-P-1})].$$

Now to prove the theorem it necessary only to show, that $[(AX^M), (AX^N)] = 0$, for all $N \in \mathbf{Z}_+$, and for $M = 0, 1$. For $M = 0$ it is evident, because in this case $(AX^M) = \mathrm{Tr}A$. Validity of this when $M = 1$ follows from the fact, that $(AX) \in g_A$. That proves the theorem.

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5   Proof of Theorem 2

Part (ii) of Theorem 2 is proved like this of Theorem 1 (see the previous section). Let us prove part (i). Like in the case of the previous theorem to prove the statement it is necessary only to show that elements of the type (AX^M) and (AX^N) commute. We will use the same inductive proof. It is possible to show [8] that elements $(X^N)_{k,l}$ are tensorial:

$$[X_{i,j},(X^N)_{k,l}] = \delta_{k,j}(X^N)_{i,l} - \delta_{i,l}(X^N)_{k,j} + \epsilon_i\epsilon_j(\delta_{j,-l}(X^N)_{k,-i} - \delta_{k,-i}(X^N)_{-j,l}).$$

Besides we will need the following proposition:

Proposition 3. $(X^{M+1})_{i,j} = \sum\limits_{p=0}^{M+1} C_p \epsilon_i \epsilon_j (X^p)_{-j,-i},$

where C_p depend on the Casimir elements of the type $(X^k), k < p, C_{M+1} = (-1)^{M+1}$.

Proof . Proof of this proposition is inductive. Obviously, that it is true for $X^0_{i,j} = \delta_{i,j}$. It is true by definition for $X^1_{i,j} = X_{i,j}$. Let us suppose that it is true also for $M = M_0$. It is easy to show that

$$(X^{M_0+1})_{i,j} = -\epsilon_i\epsilon_k(X^{M_0})_{k,j}X_{-k,-i} + \mathrm{Tr}(id)(X^{M_0})_{i,j} - \epsilon_i\epsilon_j(X^{M_0})_{-j,-i} - \epsilon_i\epsilon_j(X^{M_0})(X^0)_{-j,-i}.$$

So, from the assumption that the statement of proposition is true for $M = M_0$ it follows that it is true also for $M_0 + 1$. That proves the proposition.

Proposition 4. $[(X^M)_{i,j},(X^N)_{k,l}] = \sum\limits_{S=1}^{M} ((X^{M+N-S})_{i,l}(X^{S-1})_{k,j} - (X^{S-1})_{i,l}(X^{N+M-S})_{k,j})$

$- \sum\limits_{p=0}^{N} C_p (\sum\limits_{S=1}^{M} (\epsilon_{-l}\epsilon_k(X^{M+p-S})_{i,-k}(X^{S-1})_{-l,j} - \epsilon_{-k}\epsilon_l(X^{S-1})_{i,-k}(X^{p+M-S})_{-l,j})).$

Proof .Using the Leibnitz rule we obtain that

$$[(X^M)_{i,j},(X^N)_{k,l}] = \sum\limits_{S=1}^{M} \sum\limits_{u,v\in I} (X^{S-1})_{i,u}[X_{u,v},(X^N)_{k,l}](X^{M-S})_{v,j}.$$

Using tensorial nature of elements $(X^N)_{k,l}$ we have

$$[(X^M)_{i,j}, (X^N)_{k,l}] = \sum_{S=1}^{M} ((X^{M+N-S})_{i,l}(X^{S-1})_{k,j} - (X^{S-1})_{i,l}(X^{N+M-S})_{k,j}$$

$$+ \sum_{u \in I} (\epsilon_u \epsilon_{-l}(X^{S-1})_{i,u}(X^N)_{k,-u}(X^{M-S})_{-l,j} - \epsilon_u \epsilon_{-k}(X^{S-1})_{i,-k}(X^N)_{-u,l}(X^{M-S}_{u,j})).$$

Using Proposition 3 we obtain the statement of the Proposition 4.

We will prove the theorem if we prove the equality $[(AX^K),(AX^N)] = 0$, for all $N, K \in$

$\mathbf{Z}_+$, i.e. that $\sum_{i,j,k,l \in I} A_{j,i}A_{l,k}[(X^K)_{i,j}, (X^N)_{k,l}] = 0$.

We will need also equality $\sum_{i,j,k,l \in I} A_{j,k}A_{l,i}[(X^K)_{i,j}, (X^N)_{k,l}] = 0$, which will be intercon-

nected with the first one in its inductive proof. Let's proceed inductively, assuming, that

these equalities are valid for all $K < M$ and arbitrary $N \in \mathbf{Z}_+$. We have to draw out that

these equalities are valid also for $K = M$. These assumptions will be proved by

Proposition 5. *If $A_{i,j} = \pm \epsilon_j \epsilon_{-i} A_{-j,-i}$ then the next equalities hold:*

(i) $\sum_{i,j,k,l \in I} A_{j,i}A_{l,k}[(X^M)_{i,j}, (X^N)_{k,l}] = \sum_{S=1}^{M} \sum_{i,j,k,l \in I} A_{j,i}A_{l,k}[(X^{S-1})_{i,j}, (X^{N+M-S})_{k,l}]+$

$+ \sum_{P=0}^{N} C_p (\sum_{S=1}^{M} \sum_{i,j,k,l \in I} A_{j,k}A_{l,i}[(X^{S-1})_{i,j}, (X^{P+M-S})_{k,l}]).$

(ii) $\sum_{i,j,k,l \in I} A_{j,k}A_{l,i}[(X^M)_{i,j}, (X^N)_{k,l}] = \sum_{S=1}^{M} \sum_{i,j,k,l \in I} A_{j,i}A_{l,k}[(X^{S-1})_{i,j}, (X^{N+M-S})_{k,l}]+$

$\pm \sum_{P=0}^{N} C_P \sum_{S=1}^{M} \sum_{Q=1}^{S} (C_Q \sum_{i,j,k,l \in I} A_{l,i}A_{j,k}[(X^{Q-1})_{i,j}, (X^{P+M-S})_{k,l}]).$

Proof: Let us prove part (i) of the Proposition. Transforming its left-hand-side we obtain:

$$\sum_{i,j,k,l \in I} A_{j,i}A_{l,k}[(X^M)_{i,j}, (X^N)_{k,l}] = \sum_{S=1}^{M} ((AX^{S-1}AX^{M+N-S}) - (AX^{S-1}AX^{M+N-S}))-$$

$$- \sum_{S=1}^{M} \sum_{i,j,k,l \in I} A_{j,i}A_{l,k}[(X^{S-1})_{i,l}, (X^{N+M-S})_{k,j}]+$$

$$+ \sum_{p=0}^{N} C_p (\sum_{S=1}^{M} (\epsilon_{-l}\epsilon_k A_{j,i}A_{l,k}(X^{M+p-S})_{i,-k}(X^{S-1})_{-l,j} - \epsilon_{-k}\epsilon_l A_{j,i}A_{l,k}(X^{p+M-S})_{-l,j}(X^{S-1})_{i,-k}))$$

$$- \sum_{P=0}^{N} C_p (\sum_{S=1}^{M} \sum_{i,j,k,l \in I} \epsilon_{-l}\epsilon_k A_{j,i}A_{l,k}[(X^{S-1})_{i,-k}, (X^{P+M-S})_{-l,j}]).$$

First summand is identically zero. It is easy to see that after relabelling of the indices in

the second summand $j \leftrightarrow l$ we will have it in the form of (i). If we relabell indices $i \leftrightarrow -l$,

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$j \leftrightarrow -k$ in the second part of the third summand we will have it in the following form

$$\sum_{p=0}^{N} C_p \sum_{S=1}^{M} (\epsilon_{-l}\epsilon_k A_{j,i}A_{l,k}(X^{M+p-S})_{i,-k}(X^{S-1})_{-l,j} - \epsilon_j\epsilon_{-i}A_{-i,-j}A_{-k,-l}(X^{p+M-S})_{i,-k}(X^{S-1})_{-l,j})$$

from which it is easy to see that this summand is identically zero if and only if $A_{i,j} = \pm\epsilon_j\epsilon_{-i}A_{-j,-i}$. Under this condition after relabelling the indices $j \to l, -l \to k, \ k \to -j$ it is easy to see that fourth summand coinsides with the second summand in the right-hand-side of (i). We have proved (i). By analogous direct calculations, using Proposition 3, one can verify validity of (ii). Now to prove the theorem it remains only to show that

$$\sum_{i,j,k,l\in I} A_{j,i}A_{l,k}[X_{i,j},(X^N)_{k,l}] = 0, \text{ and } \sum_{i,j,k,l\in I} A_{j,k}A_{l,i}[X_{i,j},(X^N)_{k,l}] = 0.$$

First equality follows from the fact that $(AX) \in g_A$. Let us prove the second equality. Transforming its left-hand-side we obtain

$$\sum_{i,j,k,l\in I} A_{j,k}A_{l,i}[X_{i,j},(X^N)_{k,l}] = \sum_{i,j,k,l\in I} A_{j,k}A_{l,i}((\delta_{k,j}(X^N)_{i,l} - \delta_{i,l}(X^N)_{k,j}) +$$

$$+\epsilon_i\epsilon_j(\delta_{j,-l}(X^N)_{k,-i} - \delta_{k,-i}(X^N)_{-j,l})) = \sum_{i,j,k,l\in I}(\epsilon_i\epsilon_{-l}A_{l,i}A_{-l,k}(X^N)_{k,-i} -$$

$$-\epsilon_i\epsilon_j A_{j,-i}A_{l,i}(X^N)_{-j,l}) = (A^2X^N) - (A^2X^N) = 0.$$

That completes the proof of the theorem.

6   Proof of Theorem 3.

Taking into account commutativity of the algebras F_A and C_{g_A} provided by Theorems 1-2 and the assumption of Theorem 3 we can use the result of Vinberg [7] that the degree of transcendency of any commutative subalgebra in an associative filtered Lie algebra is equal to the degree of transcendency of correspondent subalgebra in the associated graded

algebra. That is why we may consider commutative (with respect to the Poisson brackets) subalgebra in $P(g^*)$ [5] instead any commutative subalgebra in $U(g)$. Let $S^M(X) = (X^M) = \sum\limits_{i_1,\ldots i_M \in I} X_{i_1,i_2} X_{i_3,i_4} \ldots X_{i_M,i_1}$ be a Casimir function in $P(g^*)$. We still consider all classical simple algebras as a subalgebras in $gl(n)$ and denote coordinate functions on g^* by the same letters as basic elements of g. It is easy to see that commutative subagebra $F_A(g)$ coincides with the subalgebra of the integrals of the degree not higher than one in the parameters of the shift of the Mishchenko-Fomenko algebra. Let us explain its construction. Let $A \in g^*$ be any constant covector. Then one can define functions $S_A^{k,M}(X), k < M$ from the expansion: $S^M(X + \lambda A) = \sum\limits_{k=1}^{M} S_A^{k,M}(X)\lambda^k$. Let $F(A, X)$ be commutative algebra generated by all functions $S_A^{k,M}(X), k < M, M \in Z_+$. Following [7] we call it Mishchenko-Fomenko algebra. We will need the following Lemma:

Lemma 2. *Let O_{sing} be the singular semisimple coadjoint orbit in g^* consisting of the elements of g^* of the second matrix rank. Then for independent generators of the Mishchenko-Fomenko algebra restricted to this orbit could be chosen those ones having degrees not higher than 2 in the coordinates of g^*.*

Sketch of the proof [6]. Let $\{P^M(X)\}$ be a complete set of invariant functions chosen in the form of coeficients of the characteristic polinomial. Let $P_A^{k,M}(X), k < M$ be the set of "integrals" obtained by the described above procedure of the "shift of the argument". It is easy to show that $P_A^{k,M}(X)$ are linear combinations of the minor determinants of the matrix

[5] One can identify g and g^* in the semisimple case.

[6] For more detailed proof look [10].

X of the order $M - k$. Hence after restriction to the O_{sing} only those with $M - k < 3$ (i. e. those of the degree not higher than 2 in the X variables) are not zero. Statement of the proposition now follows from the fact that $\{P^M(X)\}$ and $\{S^M(X)\}$ and, hence, $\{P_A^{k,M}(X), k < M\}$ and $\{S_A^{k,M}(X), k < M\}$ are connected by a bipolinomial correspondence.

We can consider $S_A^{k,M}(X)$ as functions depending on the X and A. Let $d_X(.)$ denotes the "matrix" gradients of functions $S_A^{k,M}(X)$ with the values in g. Let $d_A(.)$ denotes the matrix gradients of functions $S_A^{k,M}(X)$ considered as functions of parameters A. Functional independence of functions in $P(g^*)$ (as functions of X) is equivalent to the linear independence of their gradients. Let $D_X(f)(\ D_A(f))$ denotes the space of matrix gradients over variables X (parameters A) of all generators of some subalgebra $f \subset F(A, X)$. There exists a kind of duality between integrals from the Mishchenko-Fomenko algebra as functions of X and A variables [9]: $d_X(S_A^{k,M}(X)) = d_A(S_X^{M-k-1,M}(A)),\ \ D_X(F(A, X)) = D_A(F(X, A))$.[7] From the first equality follows that $D_X(F_A(X)) = D_A(F_X^2(A))$, where $F_X^2(A)$ is a subalgebra in $F(X, A)$ of the first and the second order in parameters A. From the Lemma 2 follows that $D_A(F_X^2(A)) = D_A(F(X, A))$. Hence we obtain $D_X(F_A(X)) = D_A(F(X, A)) = D_X(F(A, X))$. As it was shown in [9]: $\dim(D_X(F(A, X)) \cap [A, g]) = 1/2\dim[A, g]$. Hence we obtain: $\dim(D_X(F_A(X)) \cap [A, g]) = 1/2\dim[A, g]$. Taking into account, that $[A, g] \perp g_A$, $\dim[A, g]=\dim g - \dim g_A$, and $\mathrm{ind} g_A = \mathrm{ind} g = \mathrm{rank} g$ for semisimple matrix A, we obtain the statement of the theorem.

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[7] Note that in the right-hand-side of these relations $X_{i,j}$ play the role of the parameters of the shift and $A_{i,j}$ play the role of the coordinates of the algebra.

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